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Unphysical singularities in semiclassical level density expansions for polygon billiards

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Abstract. It is pointed out that the semiclassical periodic orbit expansion for the level density of a bounded quantum system may contain singular structure not found in the exact level density. This is illustrated for a particular system, a pseudointegrable polygon billiard. We show that here the periodic orbit expansion has singular structure on all scales of energy, and takes on negative as well as positive values. However, when sufficiently smoothed the expansion bears a good resemblance to the exact smoothed level density.

1. Introduction

Recently there has been considerable interest in finding semiclassical approximations (i.e. in the limit that Planck's constant, $\hbar$, tends to zero) for the spectra of bounded conservative quantum systems (e.g. Berry 1983). Progress has centred on the density of states function $d(E)$. If $E_1, \ldots, E_j, \ldots$ are the eigenvalues of the quantum system, with degeneracies counted separately, then (Gutzwiller 1967)

$$d(E) \equiv \sum_j \delta(E - E_j) = -\frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im} \operatorname{Tr} \frac{1}{E - \hat{H} + i\varepsilon} = -\frac{\operatorname{Im}}{\pi} \int \mathrm{d}\boldsymbol{r}\, G(\boldsymbol{r}, \boldsymbol{r}; E), \tag{1}$$

where G is the outgoing Green function,

$$G(\boldsymbol{r}, \boldsymbol{r}'; E) = \lim_{\varepsilon \to 0^+} \langle \boldsymbol{r} | (E - \hat{H} + \mathrm{i}\varepsilon)^{-1} | \boldsymbol{r}' \rangle.$$

In the theories developed by Gutzwiller (1971) and Balian and Bloch (1974) the approximation to $d(E)$ takes the form of an infinite summation over all the classical periodic orbits μ, and their repetitions q. In integrable (Arnol'd 1974) and pseudointegrable (Richens and Berry 1981, hereinafter called I) systems periodic orbits are nearly all families of similar orbits. In ergodic systems they are mostly isolated. If the μth periodic orbit is a member of an $(l_\mu - 1)$-parameter family of similar periodic orbits then the semiclassical formula takes the general form (Berry 1983)

$$d(E) = \bar{d}(E) + \sum_\mu \sum_{q=1}^{\infty} \frac{A_{\mu,q}(E)}{\hbar^{(1+l_\mu)/2}} \sin\left[q\left(\frac{S_\mu(E)}{\hbar} + \alpha_\mu\right)\right]. \tag{2}$$

Here $S_\mu(E)$ is the action around the μth periodic orbit, $A_{\mu,q}(E)$ is an $\hbar$-independent amplitude factor, α_μ is a phase associated with the caustics around the μth periodic orbit, and $\bar{d}(E)$ is the average density of states,

$$d(E) = h^{-N} \int dr_1 \ldots dr_N \, dp \ldots dp_N \, \delta(E - H(\boldsymbol{r}, \boldsymbol{p})). \tag{3}$$

Such an approach gives only an *implicit* rule for the eigenvalues. That is, in order to find the semiclassical eigenvalues, the singularities of the periodic orbit expansion must be located. Therein lie some of the major practical difficulties of this method. In fact, it is not clear that (2) *in general* represents a meaningful semiclassical spectrum, i.e. if the singularities have approximately unit strength and occur at approximately the energies E_j (although it is known that $d(E)$ may be *exactly* written in the form (2) if $A_{q,\mu}$ depends on h (Balian and Bloch 1974)).

In particular cases, however, (2) has been shown to be a valid approximation. For example, in integrable systems the periodic orbit expansion yields precisely the eigenvalues obtained by torus quantisation (Einstein 1917, Berry and Tabor 1976). In the anisotropic Kepler problem, which is classically ergodic, Gutzwiller (1982), using a largely analytical procedure, has calculated the periodic orbit expansion, and located the first semiclassical eigenvalues. His results are in close agreement with variational calculations. Further evidence for the validity of the periodic orbit expansion, when the underlying classical motion is ergodic, was provided by Berry (1981), who discusses in great detail the quantisation of Sinai's billiard. In these examples, (2) appears to be valid even for moderate values of h (i.e. not only in the limit $h \to 0$) and it is hoped that this might be true in general.

In this paper we discuss an example in which the singularities of the periodic orbit expansion *do not agree*, even approximately, with the exact eigenvalues of the system.

Our example is a classically pseudointegrable polygon billiard system. In pseudointegrable systems (I) the orbits in phase space lie on invariant N-dimensional surfaces which are topologically different from the N-torus. They are therefore not integrable in the sense of Liouville–Arnol'd (Arnol'd 1974). The chosen system admits a completely analytical treatment at the semiclassical level of quantum mechanics, and we find that the periodic orbit expansion has a highly non-physical structure, involving a dense set of singularities with various strengths, both positive and negative. On the other hand, we find numerically that the *smoothed* periodic orbit expansion (Balian and Bloch 1971) provides a good approximation to the exactly computed smoothed density of states. Finally, we conjecture that such non-physical structure is typical for the periodic orbit expansion in polygon billiards.

2. Periodic orbit expansion for polygon billiards and the 'delta-grass'

We shall now describe the particular form of the periodic orbit expansion for the density of states of an arbitrary polygon billiard (see also I).

A polygon billiard is a flat polygonal enclosure in which a classical particle moves freely with specular reflections at the polygon boundary. If all the vertex angles of the polygon are rational multiples of π then such systems are nearly always pseudointegrable (I). That is, in phase space, their orbits explore two-dimensional invariant surfaces which do not have the topology of the torus (exceptions are the square and the equilateral triangle). If any angle is irrational then the motion is conjectured to be ergodic (Hobson 1975).

Quantum mechanically we wish to find the eigenvalues of the time-independent Schrödinger equation, subject to a suitable boundary condition. We shall choose the Dirichlet conditions, i.e. the wavefunction is to vanish on the boundary.

Now motion inside billiard systems depends only on the geometry of the billiards boundary, and not on the energy (except as a parameter determining the particle velocity). It is therefore convenient to use the scaled 'energy' variable (see also Balian and Bloch 1971)

$$k^2 = 2\mu E/\hbar^2. \tag{4}$$

The k^2 Green function satisfies

$$(k^2 + \nabla_r^2)G(\boldsymbol{r}, \boldsymbol{r}'; k^2) = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{5}$$

and the density of states is given by (cf (1))

$$d(k^2) = -(1/4\pi)\,\mathrm{Im}\int \mathrm{d}\boldsymbol{r}\, G(\boldsymbol{r}, \boldsymbol{r}; k^2). \tag{6}$$

The semiclassical limit is now the limit $k \to \infty$ (rather than $\hbar \to 0$).

Now when the source and field points $\boldsymbol{r}', \boldsymbol{r}$ are very close the Green function for the polygon is equal to the free space case, which in two dimensions is

$$G(\boldsymbol{r}, \boldsymbol{r}'; k^2) \overset{r\to r'}{\simeq} -\tfrac{1}{4}\mathrm{i}H_0^{(1)}(k|\boldsymbol{r} - \boldsymbol{r}'|), \tag{7}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of zero order (Abramowitz and Stegun 1964). However, there are also contributions to G due to reflections at the boundary of the polygon billiard. Semiclassically we can regard the source $\boldsymbol{r}'$ as a 'lighthouse' which illuminates the field point $\boldsymbol{r}$ both directly and via reflections. The semiclassical Green function therefore has contributions, of the form of (7), from *all* the classical trajectories from $\boldsymbol{r}'$ to $\boldsymbol{r}$ inside the polygon billiard, i.e.

$$G(\boldsymbol{r}, \boldsymbol{r}'; k^2) = -\tfrac{1}{4}\mathrm{i}\sum_j (-1)^{\nu_j} H_0^{(1)}(k\mathcal{L}_j(\boldsymbol{r}, \boldsymbol{r}')) \tag{8}$$

where j labels the classical trajectories from $\boldsymbol{r}'$ to $\boldsymbol{r}$, $\mathcal{L}_j$ is the total length of the jth such trajectory, and ν_j the number of reflections. The factor $(-1)^{\nu_j}$ ensures that G vanishes on the boundary.

The semiclassical density of states function is, using (1),

$$d(k^2) = \frac{\mathrm{Re}}{4\pi}\sum_j \int \mathrm{d}\boldsymbol{r}\, (-1)^{\nu_j} H_0^{(1)}(k\mathcal{L}_j(\boldsymbol{r}, \boldsymbol{r})). \tag{9}$$

The closed orbits are of three types. Firstly, there are the direct orbits with zero length, i.e. for which $\mathcal{L}_j(\boldsymbol{r}, \boldsymbol{r}) = 0$. These give the average density of states (Baltes and Hilf 1978)

$$\bar{d}(k^2) = A/4\pi \tag{10}$$

where A is the area enclosed by the polygon. Secondly there are the non-periodic (self-intersecting) orbits. For such orbits $\mathcal{L}_j$ varies with $\boldsymbol{r}$ and neighbouring contributions to the integrals in (9) cancel due to destructive interference between the rapid oscillations of the Hankel function in the semiclassical limit, so that these paths do not contribute to $d(k^2)$. Thirdly, there are the closed orbits which are also closed in momentum, i.e. the periodic orbits. These give rise to the dominant oscillatory contributions to $d(k^2)$. In pseudointegrable systems, as in integrable systems, the periodic orbits are not isolated but form continuous families on the invariant surfaces in phase space. Each orbit in the family has the same value of $\mathscr{L}_j$, so that the integrand in (9) is independent of $\boldsymbol{r}$. For integrable systems each family of periodic orbits fills its invariant surface (a torus). But in pseudointegrable systems the family typically fills a band only partially covering the surface. In the coordinate space this band may self-intersect. That is, a given periodic orbit may pass through a particular point $\boldsymbol{r}$ several times, with different directions, before repeating itself, thus dividing the periodic orbit into a number of legs. The contribution to the integral in (9) is correctly taken into account by including each such leg of the periodic orbit separately so that the family contributes a factor $\mathscr{A}_j$ equal to the area occupied by the band of periodic orbits on the invariant phase space surface. Finally, the number of reflections ν_j along a periodic orbit family in a polygon billiard is always even (I), so that

$$d(k^2) = \frac{A}{4\pi} + \frac{\text{Re}}{4\pi} \sum_{\mu} \sum_{q} \mathscr{A}_\mu H_0^{(1)}(kq\mathscr{L}_\mu), \tag{11}$$

where μ labels the primitive periodic orbit and q includes repetitions. The more familiar form of the periodic orbit expansion (2) is regained by taking the asymptotic form of the Hankel functions in (11), but for our purposes here it will be more convenient to use (11).

The formula (11) is the first-order semiclassical expression for $d(k^2)$ valid for an arbitrary polygon billiard. For the purpose of classifying the periodic orbit families involved in (11) it is convenient to introduce the *periodic orbit spectrum*, defined by

$$n(\mathscr{L}) = \sum_{\mu} \sum_{q=1}^{\infty} \mathscr{A}_\mu \delta(\mathscr{L} - q\mathscr{L}_\mu). \tag{12}$$

Then

$$d(k^2) - \bar{d}(k^2) = \frac{\text{Re}}{4\pi} \int_0^\infty \mathrm{d}\mathscr{L}\, n(\mathscr{L}) H_0^{(1)}(k\mathscr{L}). \tag{13}$$

No general method for calculating $n(\mathscr{L})$ for polygon billiards is yet known. However, $n(\mathscr{L})$ is known for some special cases. One of these is the truncated triangle billiard of figure 1(a) composed of three right-angled triangles. The invariant surfaces in phase space (figure 1(b)) have the topology of the two-handled sphere (I).

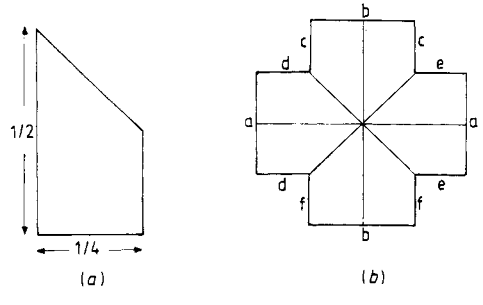

Figure 1. (a) The truncated triangle billiard and (b) the invariant phase space surface.

Table 1. The lowest 36 eigenvalues of the billiard of figure 1(*a*). The numbers shown are the 'scaled' eigenvalues k^2/π^2, which were computed accurate to the nearest integer (in I). The few exactly known eigenvalues of equation (14) are marked by an asterisk.

25	48	73	80*	106	116	150	158	160*
187	208*	211	245	263	267	272*	298	320*
326	338*	362	387	400*	410	416*	438	455
464*	476	497	508	544*	545	561	584	592*

In (I) the first 36 eigenvalues of this billiard were numerically computed (table 1). In fact, some of these eigenvalues may be obtained exactly by a generalisation of the 'torus' eigenvalue condition (I); these are

$$k_{mn}^2 = 16\pi^2(m^2+n^2), \qquad \text{where } m > n > 0. \tag{14}$$

However, the number of these eigenvalues with 'energy' less than k^2, divided by k^2, is only $1/128\pi$, where as the average density according to the area rule (10) is $3/128\pi$. It is not currently known how to find the missing eigenvalues analytically.

The periodic orbit spectrum for the truncated triangle billiard is exactly (Henyey and Pomphrey 1982, Richens 1982)

$$n(\mathscr{L}) = \sum_{q=1}^{\infty}\left(\sum_{\mu_1\geq\mu_2\geq 0}^{(E)} \tfrac{3}{4}\mathscr{B}_{\mu}\delta(\mathscr{L}-\tfrac{3}{2}q\mu) + \sum_{\mu_1\geq\mu_2\geq 0}^{(O)} [\tfrac{1}{2}\mathscr{B}_{\mu}\delta(\mathscr{L}-q\mu)+\tfrac{1}{4}\mathscr{B}_{\mu}\delta(\mathscr{L}-\tfrac{1}{2}q\mu)]\right) \tag{15}$$

where $\boldsymbol{\mu}$ is a coprime vector (i.e. the integers $\boldsymbol{\mu}=(\mu_1,\mu_2)$ have no common integer factor) and

$$\begin{aligned} \mathscr{B}_{\mu} &= 0 && \text{if } \mu_1=\mu_2=0, \\ &= \tfrac{1}{2} && \text{if } \mu_1=\mu_2 \text{ or } \mu_2=0, \\ &= 1 && \text{otherwise.} \end{aligned}$$

The notation (E) on the $\boldsymbol{\mu}$ summation means that $\mu_1+\mu_2$ is even and the notation (O) means that $\mu_1+\mu_2$ is odd. It is because this restriction is on the *coprime* vectors $\boldsymbol{\mu}$ rather than on the *lattice* vectors $\boldsymbol{\rho}=q\boldsymbol{\mu}$ that this expression is non-trivial.

Hence from (13) and (15) the periodic orbit expansion for the density of states is

$$d(k^2) = \frac{3}{128\pi} + \frac{\text{Re}}{4\pi}\sum_{q=1}^{\infty}\left(\sum_{\mu_1\geq\mu_2\geq 0}^{(E)} \tfrac{3}{4}\mathscr{B}_{\mu}H_0^{(1)}(\tfrac{3}{2}kq\mu) + \sum_{\mu_1\geq\mu_2\geq 0}^{(O)} [\tfrac{1}{2}\mathscr{B}_{\mu}H_0^{(1)}(kq\mu)+\tfrac{1}{4}\mathscr{B}_{\mu}H_0^{(1)}(\tfrac{1}{2}kq\mu)]\right). \tag{16}$$

The constant term is the average density $\bar{d}(k^2)$, (10). To the author's knowledge this is the first time such an explicit formula for the semiclassical density of states function for a non-integrable system has been written down (but see also Gutzwiller 1982).

Before analysing this expression for $d(k^2)$ we shall compare it numerically with the exact density of states (table 1). We compute the *smoothed* densities (Balian and Bloch 1971); that is, the 'energy' k^2 is given a small positive imaginary part δ. Then the exact density (1) becomes a sum of Lorentzian resonances,

$$d(k^2) = \sum_j \frac{\delta/\pi}{(k^2-k_j^2)^2+\delta^2}, \tag{17}$$

and each term in the periodic orbit summation acquires an exponential 'damping' factor $\exp(-\delta q\mathscr{L}_i|2k)$. This comparison is shown in figure 2 for $\delta = 7.5\pi^2$. This agreement is clearly quite good. However, when δ is reduced to π^2 the semiclassical curve begins to develop a striking structure of 'spikes', of both positive and negative values (figure 3), which does not correspond to any structure found in the exact density of states (1). In fact, we shall see shortly that if δ is further reduced this structure becomes more and more complicated.

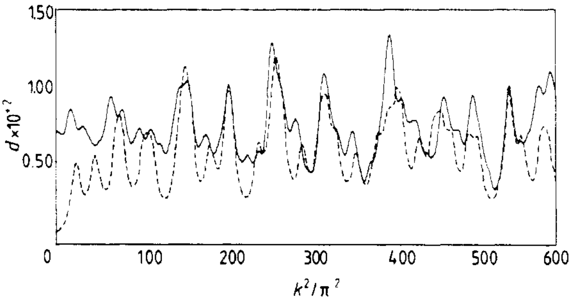

Figure 2. Smoothed semiclassical expansion of the density of states for the truncated triangle when $\delta = 7.5\pi^2$ (full curve), compared with the exact smoothed density of states (broken curve).

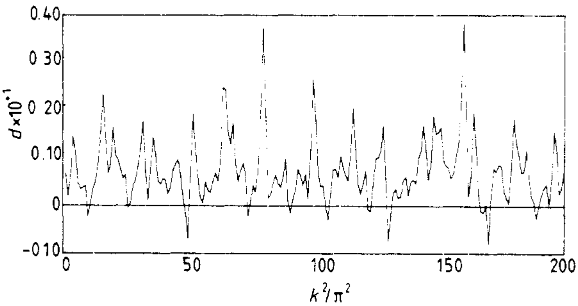

Figure 3. 'Delta-grass': the semiclassical expansion for the density of states of the truncated triangle when $\delta = \pi^2$.

We shall investigate the structure of (16) analytically by rewriting it as a summation involving δ-functions (cf (1)). The standard way of achieving this is by using the Poisson transformation technique. This re-expresses a lattice summation (i.e. one not restricted by a condition such as (E) or (O) above) as a summation over the *dual* lattice. To this end, (16) must be rewritten in a way that does not involve the restrictions (E) and (O) on the *primitive* lattice vectors $\boldsymbol{\mu}$.

To this end, we note that every lattice vector $\boldsymbol{\rho}$ whose primitive vector $\boldsymbol{\mu}$ satisfies the condition (E) (excepting the zero vector) is found uniquely amongst the vectors

$$\boldsymbol{\rho} = 2^g(2\nu_1+1, 2\nu_2+1), \tag{18}$$

where $g = 0, 1, 2, 3, \ldots$ and ν_1, ν_2 are integers. This is because $\mu_1+\mu_2$ is even if and only if the prime decompositions of ρ_1 and ρ_2 contain the same power of 2. Because of the factor 2^g in (18) the array of such vectors has a scaling or hierarchical structure (figure 4).

Using this result we may rewrite any lattice sum restricted by the condition that $\mu_1+\mu_2$ is even as

$$\sum_{q=1}^{\infty}\sum_{\boldsymbol{\mu}}{}^{(\mathrm{E})} f(q\boldsymbol{\mu}) = \sum_{g=0}^{\infty}\sum_{\boldsymbol{\rho}}[f(2^{g+1}\boldsymbol{\rho}/\sqrt{2})-f(2^{g+1}\boldsymbol{\rho})] \tag{19}$$

where $f(\boldsymbol{\rho})$ is any smooth function. This formula may be used to rewrite (16) in terms of unrestricted lattice summations of the form

$$\mathrm{Re}\sum_{\rho_1\geq\rho_2\geq 0}\mathscr{B}_{\boldsymbol{\rho}} H_0^{(1)}(\alpha k\rho). \tag{20}$$

The Poisson transformation of this unrestricted lattice summation yields the exact result,

$$\sum_{M_1\geq M_2\geq 0}\mathscr{B}_{\boldsymbol{M}}\delta\left(k^2-\frac{4\pi^2}{\alpha^2}\boldsymbol{M}^2\right)=\frac{\alpha^2}{32\pi}+\frac{\alpha^2}{4\pi}\mathrm{Re}\sum_{\rho_1\geq\rho_2\geq 0}\mathscr{B}_{\boldsymbol{\rho}} H_0^{(1)}(\alpha k\rho). \tag{21}$$

The left-hand side is the density of states function for a $\pi/4$ right triangular billiard, and the right-hand side its expansion over periodic orbits, which in this case is exact.

Putting together (16), (19) and (21) we find, after some rearrangement, the following curious semiclassical expression for the density of states of our polygon billiard:

$$\begin{aligned} d(k^2) = &\sum_{M_1\geq M_2\geq 0}\mathscr{B}_{\boldsymbol{M}}[\delta(k^2-16\pi^2\boldsymbol{M}^2)+\delta(k^2-8\pi^2\boldsymbol{M}^2)] \\ &+\sum_{g=0}^{\infty}\frac{1}{4^g}\sum_{M_1\geq M_2\geq 0}\mathscr{B}_{\boldsymbol{M}}\left\{\tfrac{1}{12}\left[\delta\left(k^2-\frac{8\pi^2\boldsymbol{M}^2}{9\times 4^g}\right)-2\delta\left(k^2-\frac{4\pi^2\boldsymbol{M}^2}{9\times 4^g}\right)\right]\right. \\ &\left.+\tfrac{3}{4}[\delta(k^2-4\pi^2\boldsymbol{M}^2/4^g)-2\delta(k^2-8\pi^2\boldsymbol{M}^2/4^g)]\right\}. \end{aligned} \tag{22}$$

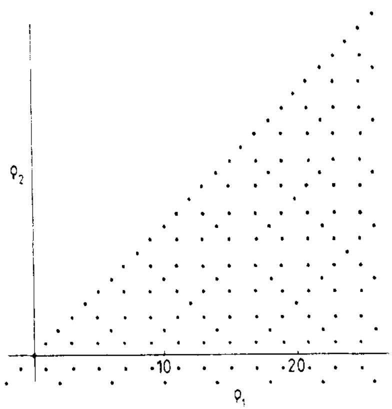

Figure 4. Square lattice points $\boldsymbol{\rho}=q\boldsymbol{\mu}$ (where $\boldsymbol{\mu}$ is a coprime vector) restricted by the condition that $\mu_1+\mu_2$ is even.

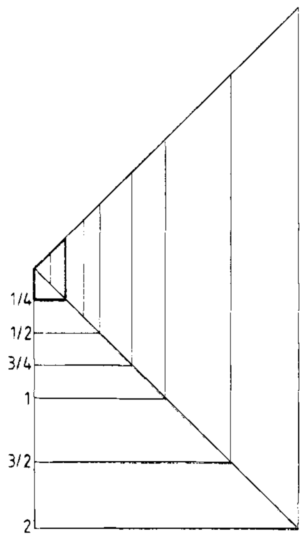

Figure 5. $\pi/4$ right triangular billiards involved in the semiclassical expansion (24) for the density of states of the truncated triangle. The original billiard of figure 1(a) is indicated by the thick lines.

This is *exactly* equal to the periodic orbit summation (16). The first two $\boldsymbol{M}$-summations have an average density of $3/128\pi$, which is the correct average density for this billiard. The first of these two terms are the exact eigenvalues of equation (14). The second of these two terms together with the g-summation are the first-order semiclassical expression for the remaining eigenvalues (for which exact analytical results are not known). The g-summation has an average value of zero. The δ-functions in the g-summation are spread *densely* over the $\boldsymbol{k}$-plane, and take both positive and negative values. For obvious reasons we have called the expression (22) a *delta-grass*. Clearly it bears no obvious relationship to the exact formula, (1); we shall discuss this further in § 3.

It is instructive to rewrite (22) using the function

$$\Lambda(L;k) = (8\pi/L^2) \sum_{M_1 \geqslant M_2 \geqslant 0} \mathscr{B}_{\boldsymbol{M}} \delta(k^2 - \pi^2 \boldsymbol{M}^2/L^2). \tag{23}$$

This is proportional to the density of states of the $\pi/4$ right triangle in which the short sides have length L. Λ is normalised to have an average density of unity. In terms of this function (22) becomes

$$d(k^2) = \frac{1}{128\pi}\Lambda(\tfrac{1}{4};k) + \frac{1}{64\pi}\Lambda\left(\frac{1}{2\sqrt{2}};k\right) + \frac{3}{128\pi}\sum_{g=0}^{\infty}\left[\Lambda\left(\frac{3\times 2^g}{2\sqrt{2}};k\right) - \Lambda\left(\frac{2^g}{2\sqrt{2}};k\right) + \Lambda\left(\frac{2^g}{2};k\right) - \Lambda\left(\frac{3\times 2^g}{2};k\right)\right]. \tag{24}$$

This is an infinite expansion for the density of states in the truncated triangle involving the density of states functions of larger and larger $\pi/4$ right triangle billiards. The geometrical relationship between these billiards is shown in figure 5.

3. Discussion

We have just seen that the semiclassical periodic orbit expansion for the density of states of the 'truncated triangle' billiard of figure 1 has a complicated 'δ-grass' structure, which bears no obvious relationship to the exact density of states (1). However, when k^2 has a small positive imaginary part, so that these δ-functions become broadened into Lorentzian resonances, much of this structure is smoothed away and it is for this reason that the *smoothed* exact and *smoothed* semiclassical densities of states functions are in good agreement (figure 2).

We have found that the semiclassical density of states has an interpretation as an infinite sum over the density of states functions of larger and larger $\pi/4$ right triangular billiards (24). Now the functions $\Lambda(L;k)$, (23), obey the scaling law

$$\Lambda(L;k) = \Lambda(\lambda L; k/\lambda), \tag{25}$$

which means that the terms S_g in the g-summation (24) obey the scaling law

$$S_g(k) = S_{g+n}(k/2^n). \tag{26}$$

This suggests the possibility that the expansion (24) might be obtainable using a renormalisation theory, in a way not yet known.

Now the eigenvalues of the $\pi/4$ right triangular billiard are highly degenerate; in fact the levels have an average degeneracy proportional to $\sqrt{\ln k}$ (Berry 1981). Hence the 'δ-grass' will be 'thinned out' at this rate as $k \to \infty$. However, because of the scaling law (26) there will always remain unphysical structure in the periodic orbit expansion, even as $k \to \infty$. In fact, it appears that the 'δ-grass' may only be removed by a more exact treatment of the density of states function.

A careful analysis of the Green function inside polygon billiards shows that apart from the contributions associated with the classical bands of orbits (11), there are also contributions associated with orbits scattered at the *vertices* of the polygon (Richens 1982, Balian and Bloch 1971). Semiclassically, there is a contribution for each *scattering route* consisting of straight line paths linking successive vertices of the polygon (and possibly including specular reflections at the polygon edges). The contribution from each *periodic* scattering route to the semiclassical density of states function is of order $1/k^{(\nu+2)/2}$, where ν is the number of vertex scatterings along the route (Richens 1982). That is, each vertex reduces the asymptotic order by $1/k^{1/2}$. The number of such contributions increases *exponentially* with their length. There is thus a delicate balance between this proliferation of routes and their increasingly smaller asymptotic values, in determining their *overall* contribution to $d(k^2)$. A treatment which takes *exact* account of these contributions will, of course, lead to an *exact* representation of the density of states function, and thus resolve the difficulties associated with the periodic orbit expansion. This remains a difficult problem for the future.

It is interesting to speculate on whether the 'δ-grass' structure of the periodic orbit expansion for this particular polygon billiard system is a typical property for general polygon billiards. This seems likely (at least for finite values of k) for the following reason. In the 'space of all conceivable polygons' there is a dense set (of zero measure) of polygons for which the periodic orbit vectors lie on a regular lattice, restricted by a condition on the primitive lattice vectors (as in our example, i.e. figure 4). For these polygons an analysis similar in principle to the one carried out in § 2 will lead directly to a 'δ-grass' formula analogous to (22). Any other polygon may be approximated arbitrarily closely by one of these special cases, so that, at least for finite k, the 'δ-grass' will persist. We therefore conjecture that the 'δ-grass' is a generic feature of the periodic orbit expansion for the density of states in polygon billiard systems. Of course, such curious behaviour is not expected to be typical of periodic orbit expansion *in general*, but is here due to the *singular* nature of the classical flow at the vertices of the polygon billiards.

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